

# On the class of distributions of subordinated Lévy processes

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## Abstract

This article study the class of distributions obtained by subordinating Lévy processes and Lévy bases. To do this we derive properties of a suitable mapping obtained via Lévy mixing. We show that our results can be used to solve the so-called recovery problem for general Lévy bases as well as for moving average processes which are driven by subordinated Lévy processes.

**Keywords:** Subordination, Lévy basis, Lévy processes, Lévy mixing, Lévy semistationary processes, recovery problem.

## 1 Introduction

Lévy processes and Lévy bases constitute important building blocks for constructing realistic models for temporal and or spatial phenomena. In addition, it has often been noted that *stochastic volatility/intermittency*, which can be regarded as stochastic variability beyond the fluctuations described by the Lévy noise, is present in empirical data of e.g. asset prices in finance or turbulence in physics. One possibility of accounting for such additional variability is by using the concept of subordination, see Clark (1973), Monroe (1978) and also Veraart and Winkel (2010) for additional references. In this article, we are interested in characterizing the law of subordinated Lévy processes and Lévy bases. We will approach this problem from the general angle of defining a mapping

$$\Phi_M(\rho)(A) := \int_0^\infty \mu(s, A) \rho(ds), \quad A \in \mathcal{B}(\mathbb{R} \setminus \{0\}), \quad (1)$$

where  $M = (\mu(s, \cdot))_{s \geq 0}$  denotes a measurable collection of probability measures and  $\rho$  is a Lévy measure on  $\mathbb{R}^+$ . We note that if  $\mu(s, \cdot)$  is chosen as the law of a Lévy process at time  $s$  and  $\rho$  is a  $\sigma$ -finite measure supported on the positive half line, then  $\Phi_M$  describes the Lévy measure of a subordinated Lévy process. We will discuss in which sense the mapping is related to the concept of *Lévy mixing*, as introduced by Barndorff-Nielsen et al. (2013b).

While the mapping (1) can be defined for various measures  $\mu$ , we are mainly interested in the situation when  $\mu(s, dx) = \mu^s(dx)$  for an infinitely divisible (ID) law  $\mu$ . In that case, we shorten the notation and typically write  $\Phi_\mu = \Phi_M$ . In particular, we focus on three scenarios in more detail: The cases when 1.)  $\Phi_\mu$  is restricted to the finite Lévy measures on  $(0, \infty)$ , 2.)  $\mu$  is symmetric, and 3.)  $\mu$  is supported on  $(0, \infty)$ . Our first results in this context contain a detailed description of the properties of the mapping  $\Phi_\mu$ . In particular, we characterize its Lévy domain and some properties of its range as well as establish conditions under which the mapping  $\Phi_\mu$  is one-to-one.

These results can then be used to describe the law of subordinated Lévy processes and bases.

As an application of our results, we study the so-called *recovery problem* for subordinated ID processes: If we observe a subordinated Lévy process  $X_t = L_{T_t}$  (where  $L$  is a Lévy process and  $T$

is an independent subordinator), can we recover  $T$  from  $X$  and, if so, in which sense? In answering this question we will build upon and further extend earlier work by Winkel (2001) and Geman et al. (2002).

Moreover, we will go one step further and use such a subordinated Lévy process as the driving noise in a moving average type process and derive suitable conditions which allow us to recover the subordinator from observations of the moving average process. In the special case of a Brownian moving average process restricted to be a semimartingale, such a problem has been addressed by Comte and Renault (1996). More recently, Sauri (2015) studied the invertibility of infinitely divisible continuous moving averages processes and we can build on this result to solve the recovery problem in the more general set-up which includes non-semimartingale processes.

The outline for the remaining article is as follows. Section 2 introduces the basic notation and background material on Lévy processes, Lévy bases, subordination and meta-times. In Section 3, we will define the mapping (1) and study its key properties. We will then use these results in Section 4 to describe the law of subordinated Lévy processes and bases. The recovery problem for moving average processes driven by subordinated Lévy processes is then studied in Section 5.

## 2 Preliminaries and basic results

Throughout this article  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}}, \mathbb{P})$  denotes a filtered probability space satisfying the usual conditions of right-continuity and completeness. A two-sided Lévy process  $(L_t)_{t \in \mathbb{R}}$  on  $(\Omega, \mathcal{F}, \mathbb{P})$  is a real-valued stochastic process taking values in  $\mathbb{R}$  with independent and stationary increments whose sample paths are almost surely càdlàg. If we drop the stationarity in the increments of  $(L_t)_{t \in \mathbb{R}}$ , we call it an additive processes. We say that  $(L_t)_{t \in \mathbb{R}}$  is an  $(\mathcal{F}_t)$ -Lévy process if for all  $t > s$ ,  $L_t - L_s$  is  $\mathcal{F}_t$ -measurable and independent of  $\mathcal{F}_s$ .

Denote by  $ID(\mathbb{R})$  the space of probability measures on  $\mathbb{R}$  that are infinitely divisible. The law of a Lévy process is in a bijection with  $ID(\mathbb{R})$  and  $L_1$  has a Lévy-Khintchine representation given by

$$\log \widehat{\mu}(z) = iz\gamma - \frac{1}{2}z^2b + \int_{\mathbb{R}^d} [e^{izx} - 1 - i\tau(x)z] \nu(dx), \quad z \in \mathbb{R},$$

where  $\widehat{\mu}$  is the characteristic function of the distribution of  $L_1$ ,  $\gamma \in \mathbb{R}$ ,  $b \geq 0$  and  $\nu$  is a Lévy measure, i.e.  $\nu(\{0\}) = 0$  and  $\int_{\mathbb{R}} (1 \wedge |x|^2) \nu(dx) < \infty$ . Here, we assume that the truncation function  $\tau$  is given by  $\tau(x) = x \mathbf{1}_{\{|x| \leq 1\}}$ , for  $x \in \mathbb{R}$ . When  $\mu$  has support on  $(0, \infty)$ , we write  $\mu \in ID(\mathbb{R}^+)$ . We say that  $\mu \in ID(\mathbb{R})$  is strictly  $\alpha$ -stable if for any  $s > 0$  and  $z \in \mathbb{R}$ ,  $\widehat{\mu}(z)^s = \widehat{\mu}(s^{1/\alpha}z)$ . If  $\mu \in ID(\mathbb{R})$  has characteristic triplet  $(\gamma, b, \nu)$  satisfying  $\int_{\mathbb{R}} (1 \wedge |x|) \nu(dx) < \infty$ , we use the triplet relative to the truncation function  $\tau_0(x) = 0$ , and we write  $(\beta_0, b, \nu)_0$ . Here  $\beta_0 = \gamma - \int_{|x| \leq 1} x \nu(dx)$ . Moreover, in the same context, if  $b = 0$ , we will write  $(\beta_0, \nu)_0$  instead of  $(\beta_0, 0, \nu)_0$  and we refer to it as the characteristic pair.

By *infinitely divisible continuous time moving average* (IDCMA) process we mean a real-valued stochastic process  $(X_t)_{t \in \mathbb{R}}$  on  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}}, \mathbb{P})$  given by the following formula

$$X_t := \theta + \int_{\mathbb{R}} f(t-s) dL_s, \quad t \in \mathbb{R}. \quad (2)$$

where  $\theta \in \mathbb{R}$ ,  $f$  is a deterministic function and  $L$  is a Lévy process with triplet  $(\gamma, b, \nu)$ . Observe that  $X$  is strictly stationary and infinitely divisible in the sense of Barndorff-Nielsen et al. (2006) and Barndorff-Nielsen et al. (2014).

A *Lévy semistationary process* ( $\mathcal{LSS}$ ) on  $\mathbb{R}$  is a stochastic process  $(Y_t)_{t \in \mathbb{R}}$  on  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}}, \mathbb{P})$  which is described by the following dynamics

$$Y_t = \theta + \int_{-\infty}^t g(t-s) \sigma_s dL_s + \int_{-\infty}^t q(t-s) a_s ds, \quad t \in \mathbb{R}, \quad (3)$$

where  $\theta \in \mathbb{R}$ ,  $L$  is a Lévy process with triplet  $(\gamma, b, \nu)$ ,  $g$  and  $q$  are deterministic functions such that  $g(x) = q(x) = 0$  for  $x \leq 0$ , and  $\sigma$  and  $a$  are adapted càdlàg processes with  $\sigma$  predictable. When  $L$  is a two-sided Brownian motion,  $Y$  is called a *Brownian semistationary process* ( $\mathcal{BSS}$ ). For further references to the theory and applications of Lévy semistationary processes, see for instance Veraart and Veraart (2014) and Benth et al. (2014). See also Brockwell et al. (2013). In the absence of a drift and a stochastic volatility component, an  $\mathcal{LSS}$  process is an IDCMA process.

## 2.1 Lévy basis

In this part we present the results of a time change of a Lévy basis that are going to be used to study the marginal distribution of a subordinated Lévy basis. We refer to Barndorff-Nielsen and Pedersen (2012) for more details on the background material.

Let  $\mathcal{S}$  be a non-empty set and  $\mathcal{R}$  a  $\delta$ -ring of subsets of  $\mathcal{S}$  having an increasing sequence  $\{S_n\} \subset \mathcal{S}$  converging to  $\mathcal{S}$ . A stochastic field  $L = \{L(A) : A \in \mathcal{R}\}$  defined on  $(\Omega, \mathcal{F}, \mathbb{P})$  is called *independently scattered random measure* (i.s.r.m. for short), if for every sequence  $\{A_n\}_{n \geq 1}$  of disjoint sets in  $\mathcal{R}$ , the random variables  $(L(A_n))_{n \geq 1}$  are independent, and if  $\cup_{n \geq 1} A_n$  belongs to  $\mathcal{R}$ , then we also have

$$L\left(\bigcup_{n \geq 1} A_n\right) = \sum_{n \geq 1} L(A_n), \quad \text{a.s.},$$

where the series is assumed to converge almost surely. A Lévy basis is an i.s.r.m. for which the law of  $L(A)$  belongs to  $ID(\mathbb{R})$  for any  $A \in \mathcal{R}$ . Any Lévy basis admits a Lévy-Khintchine representation:

$$C\{z \sharp L(A)\} = \int_A \psi(z, s) c(ds), \quad z \in \mathbb{R}, A \in \mathcal{R},$$

where  $C\{\theta \sharp X\}$  denotes the cumulant function of a random variable  $X$  and

$$\psi(z, s) := iz\gamma(s) - \frac{1}{2}b(s)z^2 + \int_{\mathbb{R}} [e^{izx} - 1 - i\tau(x)z] \nu(s, dx), \quad z \in \mathbb{R}, s \in \mathcal{S}. \quad (4)$$

The functions  $\gamma, b$  and  $\nu(\cdot, dx)$  are  $(\mathcal{S}, \mathcal{B}_{\mathcal{S}}) / (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ -measurable (in which  $\mathcal{B}_{\mathcal{S}} := \sigma(\mathcal{R})$ ) with  $b$  being non-negative and  $\nu(s, \cdot)$  is a Lévy measure for every  $s \in \mathcal{S}$ . The measure  $c$  is defined on  $(\mathcal{S}, \mathcal{B}_{\mathcal{S}})$  and is called the *control measure* of  $L$ . The quadruplet  $(\gamma, b, \nu, c)$  characterizes uniquely the distribution of  $L$  and for this reason it will be called the *characteristic quadruplet* of  $L$ . Define  $L' = (L'(s))_{s \in \mathcal{S}}$  the collection of infinitely divisible random variables in  $\mathbb{R}$  for which  $L'(s)$  has characteristic triplet  $(\gamma(s), b(s), \nu(s, \cdot))$  for any  $s \in \mathcal{S}$ . The collection  $L'$  is known as Lévy seeds and they satisfy that  $\psi(z, s) = C\{z \sharp L'(s)\}$  for all  $s \in \mathcal{S}$ .

When  $\gamma, b$  and  $\nu(\cdot, dx)$  do not depend on  $s$  and  $c$  is the Lebesgue measure (up to a constant),  $L$  is termed as *homogeneous*. In this case we say that  $L$  has triplet  $(\gamma, b, \nu)$ . In the case that  $\mathcal{S} = \mathbb{R}^k$ , homogeneity is equivalent to saying that  $L(A + x) \stackrel{d}{=} L(A)$  for any  $x \in \mathbb{R}^k$ . Note that when  $L$  is homogeneous in  $\mathbb{R}^k$  with triplet  $(\gamma, b, \nu)$ , then

$$\mathcal{L}[L(A)] = \mu^{Leb(A)}, \quad A \in \mathcal{B}_b(\mathbb{R}^k), \quad (5)$$

where  $\mathcal{B}_b(\mathbb{R}^k)$  denotes the bounded Borel sets on  $\mathbb{R}^k$  and  $\mu$  is the infinite divisible law associated with the triplet  $(\gamma, b, \nu)$ .

If we add the extra condition that  $L(\{x\}) = 0$  a.s. for all  $x \in \mathbb{R}^k$ , then  $L$  has a Lévy-Itô decomposition: We have that almost surely

$$L(A) = \int_A \gamma(s) c(ds) + W(A) + \int_A \int_{|x| > 1} x N(dx ds) + \int_A \int_{|x| \leq 1} x \tilde{N}(dx ds), \quad A \in \mathcal{B}_b(\mathbb{R}^k), \quad (6)$$

where  $W$  is a centered Gaussian process with  $\mathbb{E}[W(A)W(B)] = \int_{A \cap B} b(s) c(ds)$  for all  $A, B \in \mathcal{R}$ ,  $\tilde{N}$  and  $N$  are compensated and non-compensated Poisson random measures on  $\mathbb{R}^k \times \mathbb{R}$  with intensity  $\nu(s, dx) c(ds)$ , respectively. Additionally,  $W$  and  $N$  are independent. See Pedersen (2003) for more details.

In the homogeneous case, when  $\mathcal{S} = \mathbb{R}$ , the process  $L_t - L_s := L((s, t])$  is an  $(\mathcal{F}_t)$ -Lévy process. Reciprocally, if  $(L_t)_{t \in \mathbb{R}}$  is a Lévy process on  $\mathbb{R}$ , the random measure characterized by  $L((s, t]) := L_t - L_s$  for  $s \leq t$  is an homogeneous Lévy basis on  $\mathbb{R}$ . Therefore, homogeneous Lévy bases and Lévy processes are in a bijection. More generally, Lévy bases on  $\mathbb{R}$  are in a bijection with *natural additive processes*. See for instance Sato (2004). For the multiparameter case, i.e.  $\mathcal{S} = \mathbb{R}^k$  the same result holds but for the so-called *lamp processes*. See Pedersen (2003) and Adler et al. (1983) for more detail about these type of processes.

In analogy to subordinators on  $\mathbb{R}^+$ , let us consider the following Lévy basis

$$T(A) = \int_A \beta_0(s) c(ds) + \int_A \int_0^\infty x N(dx ds), \quad A \in \mathcal{B}_b(\mathbb{R}^k), \quad (7)$$

with  $\beta_0 \geq 0$ ,  $c$  being a continuous  $\sigma$ -finite measure on  $\mathcal{B}(\mathbb{R}^k)$  and  $N$  a Poisson random measure with intensity  $\nu(s, dx) c(ds)$ . The measure  $\nu$  satisfies that for  $c$ -almost all  $s \in \mathbb{R}^k$ ,  $\nu(s, (-\infty, 0)) = 0$  and  $\int_0^\infty (1 \wedge x) \nu(s, dx) < \infty$ . Observe that  $T$  is non-negative and for almost all  $\omega \in \Omega$ ,  $T(\omega, \cdot)$  is in fact a true measure.

From Adler et al. (1983) and Pedersen (2003),  $L$  is almost surely a non-negative measure-valued field if and only if the previously stated conditions hold. Being more precise, if  $T$  is a Lévy basis on  $\mathbb{R}^k$  with characteristic quadruplet  $(\beta, b, \rho, c)$  where  $c$  is continuous, then for almost all  $\omega \in \Omega$ ,  $T(\omega, \cdot)$  is a true measure if and only if the following conditions hold for  $c$ -almost all  $s \in \mathbb{R}^k$

1.  $b(s) = 0$ ;
2.  $\nu(s, (-\infty, 0)) = 0$ ;
3.  $\int_0^\infty (1 \wedge x) \nu(s, dx) < \infty$ ;
4.  $\beta(s) - \int_0^\infty x \rho(s, dx) \geq 0$ .

In this case  $T$  admits the representation given in (7) with  $\beta_0(s) = \beta(s) - \int_0^1 x \nu(s, dx)$ . Moreover,  $L$  can be extended to the whole  $\sigma$ -algebra  $\mathcal{B}(\mathbb{R}^k)$  by

$$T(A) := \lim_{n \rightarrow \infty} T(A \cap S_n), \quad A \in \mathcal{B}(\mathbb{R}^k),$$

where the limit exists almost surely.

## 2.2 Meta-times

Meta-times were introduced in Barndorff-Nielsen and Pedersen (2012) as a way to extend the concept of a time change of Lévy processes (see for instance Barndorff-Nielsen et al. (2006)) to the spatiotemporal case. A *meta-time*  $\varphi : \mathcal{B}_b(\mathbb{R}^k) \rightarrow \mathcal{B}_b(\mathbb{R}^k)$  is a set function that maps disjoint sets into disjoint sets and for any disjoint sequence  $(A_n)_{n \geq 1} \subset \mathcal{B}_b(\mathbb{R}^k)$  such that  $\cup_{n \geq 1} A_n \in \mathcal{B}_b(\mathbb{R}^k)$ , it holds that  $\varphi(\cup_{n \geq 1} A_n) = \cup_{n \geq 1} \varphi(A_n)$ . Note that if  $\varphi$  is a meta-time, then the random measure defined by

$$L_m(A) := L(\varphi(A)), \quad A \in \mathcal{B}_b(\mathbb{R}^k), \quad (8)$$

is a Lévy basis provided that  $L$  is Lévy basis as well. If  $L$  is homogeneous, then  $L_m$  satisfies

$$\mathcal{L}[L_m(A)] = \mu^{m(A)}, \quad A \in \mathcal{B}_b(\mathbb{R}^k), \quad (9)$$

where  $m$  stands for the measure induced by  $\varphi$ , this is  $m := Leb \circ \varphi$ . The most important result in Barndorff-Nielsen et al. (2006) allows to get the reversal of the previous property: Let  $m : \mathcal{B}(\mathbb{R}^k) \rightarrow \mathbb{R}^+$  be a measure which is finite in  $\mathcal{B}_b(\mathbb{R}^k)$  and such that  $m(\{x \in \mathbb{R}^k : x_i = 0 \text{ for some } i = 1, \dots, k\}) = 0$ . Then, there exists a measurable function (not necessarily unique)  $\phi : \mathbb{R}^k \rightarrow \mathbb{R}^k$  such that

$$m(A) = Leb(\phi^{-1}(A)), \quad A \in \mathcal{B}(\mathbb{R}^k).$$

Putting  $\varphi = \phi^{-1}$ , we see that in this case  $L_m$  can be defined as in (8), and (9) holds. Therefore, if  $(T(A))_{A \in \mathcal{B}(\mathbb{R}^k)}$  is a measure-valued stochastic field, for which almost surely

$$T(\{x \in \mathbb{R}^k : x_i = 0 \text{ for some } i = 1, \dots, k\}) = 0, \quad (10)$$

then for every  $\omega \in \Omega_0$ , with  $\Omega_0$  being the set where (10) holds and  $T$  is a true measure, there exists a measurable mapping  $\phi_\omega : \mathbb{R}^k \rightarrow \mathbb{R}^k$  such that

$$T(\omega, A) = Leb(\phi_\omega^{-1}(A)), \quad A \in \mathcal{B}(\mathbb{R}^k).$$

Furthermore, if  $T$  is independent of  $L$ , then we define  $L_T$  to be the *subordinated Lévy basis* by  $T$  as the random measure defined as

$$L_T(\omega, A) := L(\omega, \varphi(\omega, A)), \quad \omega \in \Omega_0, A \in \mathcal{B}_b(\mathbb{R}^k), \quad (11)$$

and an arbitrary value outside of  $\Omega_0$ . In the previous equation,  $\varphi(\omega, A) = \phi_\omega^{-1}(A)$  is the meta-time induced by  $T(\omega, \cdot)$ . Note that by equation (9), if  $L$  is an homogeneous Lévy basis, then

$$\mathcal{L}[L_T(A) | T] = \mu^{T(A)}, \quad A \in \mathcal{B}_b(\mathbb{R}^k). \quad (12)$$

However, in general,  $L_T$  is not a Lévy basis: Put  $L = Leb$  and

$$T(A) := U Leb(A), \quad U \sim \text{Uniform}[0, 1].$$

Then  $T(\omega, \cdot)$  is a true measure satisfying (10), but  $L_T$  is not infinitely divisible. Indeed, observe that in this case

$$L_T(A) = T(A), \quad A \in \mathcal{B}_b(\mathbb{R}^k).$$

which is not infinitely divisible. Nevertheless, if  $T$  is a Lévy basis, with minor changes in the proof of Theorem 5.1 in Barndorff-Nielsen and Pedersen (2012), we get that:

**Theorem 1 (Barndorff-Nielsen and Pedersen (2012))** *Let  $L$  be an homogeneous Lévy basis with characteristic triplet  $(\gamma, b, \nu)$  and  $T$  a non-negative Lévy basis with characteristic quadruplet  $(\gamma, 0, \rho, c)$ . Then  $L_T$  as in (11) is a Lévy basis with characteristic quadruplet  $(\bar{\gamma}, \bar{b}, \bar{\nu}, \bar{c})$  given by*

1.  $\bar{c} = c$ ;
2.  $\bar{\gamma}(s) = \gamma\beta_0(s) + \int_0^\infty \int_{|x| \leq 1} x\mu^r(dx) \rho(s, dr)$ ;
3.  $\bar{b}(s) = b\beta_0(s)$ ;
4.  $\bar{\nu}(s, dx) = \beta_0(s) \nu(dx) + \int_0^\infty \mu^r(dx) \rho(s, dr)$ ,

where  $\mu$  is the ID law associated to  $(\gamma, b, \nu)$  and  $\beta_0(s) = \gamma(s) - \int_0^\infty x\rho(s, dx)$ .

**Remark 2** The previous result is the generalization of the corresponding law of a subordinated Lévy process. See Theorem 30.1 in Sato (1999) for more details. Hence, if  $T$  is homogeneous, the triplet associated to  $L_T$  corresponds to the law of a subordinated Lévy process on the real line.

### 3 A class of Lévy measures obtained by Lévy mixing

In this part we study a certain mapping obtained by mixing a family of probability distributions through a Lévy measure.

Let  $\mathfrak{M}_L(\mathbb{R})$  denote the class of Lévy measures on  $\mathbb{R}$ , i.e. a  $\sigma$ -finite measure  $\rho$  on  $\mathcal{B}(\mathbb{R})$  belongs to  $\mathfrak{M}_L(\mathbb{R})$  if  $\rho(\{0\}) = 0$  and  $\int_{\mathbb{R}} (1 \wedge |x|^2) \rho(dx) < \infty$ . By  $\mathfrak{M}_L^1(\mathbb{R}) \subset \mathfrak{M}_L(\mathbb{R})$  and  $\mathfrak{M}_L^0(\mathbb{R}) \subset \mathfrak{M}_L(\mathbb{R})$  we mean the subclasses of Lévy measures satisfying  $\int_{\mathbb{R}} (1 \wedge |x|) \rho(dx) < \infty$  and  $\rho(\mathbb{R}) < \infty$ , respectively. We define in a similar way  $\mathfrak{M}_L(\mathbb{R}^+)$ ,  $\mathfrak{M}_L^1(\mathbb{R}^+)$  and  $\mathfrak{M}_L^0(\mathbb{R}^+)$ . Consider a measurable collection of probability measures  $M = (\mu(s, \cdot))_{s \geq 0}$ . We introduce the following mapping

$$\Phi_M(\rho)(A) := \int_0^\infty \mu(s, A) \rho(ds), \quad \rho \in \mathfrak{M}_L(\mathbb{R}^+), A \in \mathcal{B}(\mathbb{R} \setminus \{0\}),$$

and  $\Phi_M(\rho)(\{0\}) = 0$ . If  $\Phi_M(\rho) \in \mathfrak{M}_L(\mathbb{R})$ , then it belongs to the subclass of Lévy measures appearing by *Lévy mixing*. See Barndorff-Nielsen et al. (2013b) for more examples of Lévy mixing.

Note that when  $\mu(s, \cdot)$  is the law of a Lévy process at time  $s$ ,  $\Phi_M$  corresponds to the Lévy measure of a subordinated Lévy process by a subordinator having no drift and Lévy measure  $\rho$ . In the same context,  $\Phi_M$  also describes the mixed probability distribution obtained by subordinating a Lévy process through a random time. Being more precise, if  $\mu(s, \cdot)$  is the law of a Lévy process  $L$  at time  $s$  and  $\rho$  is a probability measure with support on  $\mathbb{R}^+$ , then  $\Phi_M(\rho)$  is the distribution of the random variable  $L_{T^\rho}$ , for a random time  $T^\rho \sim \rho$  and independent of  $L$ . Indeed, by independence

$$\begin{aligned} \mathbb{P}(L_{T^\rho} \in A) &= \int_0^\infty \mathbb{P}(L_s \in A) \rho(ds) \\ &= \Phi_M(\rho)(A), \quad A \in \mathcal{B}(\mathbb{R} \setminus \{0\}). \end{aligned} \tag{13}$$

However, such a mapping is not only limited to a family of infinitely divisible distributions. For instance, if we choose

$$\mu(s, dx) = \frac{1}{2\pi} (s - x^2)^{-1/2} \mathbf{1}_{(0, s^{1/2})}(x) dx, \quad s \geq 0,$$

the mapping  $\Phi_M$  is well defined on  $\mathfrak{M}_L^1(\mathbb{R}^+)$  and it describes all the Lévy measures of the so-called class  $A(\mathbb{R})$ . Moreover,  $\Phi_M$  is one-to-one and it is not an *upsilon transformation*. That is, there is no  $\sigma$ -finite measure  $\eta$  such that

$$\Phi_M(\rho)(A) = \int_0^\infty \eta(s^{-1}A) \rho(ds), \quad \rho \in \mathfrak{M}_L(\mathbb{R}^+), A \in \mathcal{B}(\mathbb{R} \setminus \{0\}),$$

see Arizmendi et al. (2010) and Maejima et al. (2012) for more details on these aspects as well as Maejima et al. (2013) for generalizations. For a comprehensive introduction to *upsilon transformations* we refer to Barndorff-Nielsen et al. (2008). It is remarkable that in this case  $\mu(s, \cdot)$  is not infinitely divisible in the classical sense, but according to Franz and Muraki (2004),  $\mu(s, \cdot)$  is infinitely divisible with respect to the *monotone convolution*. In fact, such a distribution plays the role of the Gaussian distribution under this operation. In contrast, if

$$\mu(s, dx) = \frac{1}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}} dx, \quad s \geq 0,$$

the induced mapping  $\Phi_M$ , as we show later, is actually equivalent to an *upsilon transformation*, it is one-to-one and corresponds to the Lévy measure of a subordinated Brownian motion.

All the examples above show the importance of  $\Phi_M$ . Hence, in this section we study some properties of this transformation including its injectivity.

In this paper, we are interested in the case when  $\mu(s, dx) = \mu^s(dx)$  for  $\mu \in ID(\mathbb{R})$ . In this context, we will write  $\Phi_\mu$  instead of  $\Phi_M$ . We focus on the following three cases:

1. The restriction of  $\Phi_\mu$  to  $\mathfrak{M}_L^0(\mathbb{R}^+)$ ;
2.  $\mu$  is symmetric;
3.  $\mu$  having support on  $[0, \infty)$ .

Surprisingly, the one-to-one property of  $\Phi_\mu$  in cases 2 and 3, as we show later, is equivalent to the case 1. We start by describing the domain and the range of  $\Phi_\mu$ .

### 3.1 The domain and the range of $\Phi_\mu$

For a given  $\mu \in ID(\mathbb{R})$ , consider the Lévy-domain  $\Phi_\mu$ , denoted by

$$\mathcal{D}_L(\Phi_\mu) := \{\rho \in \mathfrak{M}_L(\mathbb{R}^+) : \Phi_\mu(\rho) \in \mathfrak{M}_L(\mathbb{R})\},$$

and the range of  $\Phi_\mu$ , denoted by

$$\mathcal{R}_L(\Phi_\mu) = \{\Phi_\mu(\rho) \in \mathfrak{M}_L(\mathbb{R}) : \rho \in \mathcal{D}_L(\Phi_\mu)\}.$$

In this subsection we describe both  $\mathcal{D}_L(\Phi_\mu)$  and  $\mathcal{R}_L(\Phi_\mu)$ . In order to do that, we prepare a lemma.

**Lemma 3** *Let  $\mu \in ID(\mathbb{R})$  be a non-degenerate probability measure. We have that the following asymptotic holds*

$$\int_{\mathbb{R}} (1 \wedge |x|^2) \mu^s(dx) \sim cs, \quad \text{as } s \downarrow 0,$$

for some  $c > 0$ .

**Proof.** Let  $\mu$  be a non-degenerate probability measure on  $ID(\mathbb{R})$  with characteristic triplet  $(\gamma, b, \nu)$ . It is a well known fact that if  $s_n \downarrow 0$ , then the sequence of probability measures having characteristic triplet  $(0, 0, \frac{1}{s_n} \mu^{s_n})_0$  converges weakly to  $\mu$ . From the proof of Theorem 8.7 in Sato (1999), we get that in this case, the sequence of finite measures  $\rho_n(dx) := \frac{1}{s_n} (1 \wedge |x|^2) \mu^{s_n}(dx)$  converges weakly to the finite measure

$$\rho(dx) := b\delta_0(dx) + (1 \wedge |x|^2) \nu(dx).$$

In particular,  $\frac{1}{s_n} \int_{\mathbb{R}} (1 \wedge |x|^2) \mu^{s_n}(dx) \rightarrow b + \int_{\mathbb{R} \setminus \{0\}} (1 \wedge |x|^2) \nu(dx) > 0$  as  $n \rightarrow \infty$ , as required. ■

Now we proceed to describe the domain of  $\Phi_\mu$ .

**Proposition 4** *Let  $\mu \in ID(\mathbb{R})$ . Then*

1.  $\mathcal{D}_L(\Phi_\mu) = \mathfrak{M}_L^1(\mathbb{R}^+)$  provided that  $\mu$  is not a Dirac delta measure.
2. In general  $\mathcal{R}_L(\Phi_\mu) \subsetneq \mathfrak{M}_L(\mathbb{R})$ .

**Proof.** 1. Assume that  $\mu$  is not a Dirac delta measure. Let  $\overline{\nu} = \Phi_\mu(\rho)$  for some  $\rho \in \mathfrak{M}_L(\mathbb{R}^+)$ . From the previous lemma, there is  $s_0 > 0$  and constants  $C_1, C_2 > 0$  such that

$$C_1 \int_0^{s_0} s \rho(ds) \leq \int_0^{s_0} \int_{\mathbb{R}} (1 \wedge |x|^2) \mu^s(dx) \rho(ds) \leq C_2 \int_0^{s_0} s \rho(ds).$$

Therefore,  $\int_{\mathbb{R}} (1 \wedge |x|^2) \overline{\nu}(dx) < \infty$  if and only if  $\int_0^\infty (1 \wedge s) \rho(ds) < \infty$ , as required.

2. This follows by observing that if  $\mu$  has a density, then  $\Phi_\mu(\rho)$  has a density as well for any  $\rho \in \mathcal{D}_L(\Phi_\mu)$ . ■

**Remark 5** Observe that when  $\mu(dx) = \delta_{\{\gamma\}}(dx)$  for some  $\gamma \in \mathbb{R}$ , then

$$\Phi_\mu(\rho)(dx) = \begin{cases} \rho(\gamma^{-1}dx) & \text{if } \gamma \neq 0; \\ 0 & \text{otherwise.} \end{cases}$$

This means that the properties of  $\Phi_\mu$  are the properties of  $\mathfrak{M}_L(\mathbb{R}^+)$ . Hence, for the rest of the paper we will only consider non-degenerated infinitely divisible distributions.

### 3.2 The mapping $\Phi_\mu$ restricted to $\mathfrak{M}_L^0(\mathbb{R}^+)$

In this part we will show that for a given  $\mu \in ID(\mathbb{R})$ , the mapping  $\Phi_\mu$  restricted to  $\mathfrak{M}_L^0(\mathbb{R}^+)$  is one-to-one.

Before we present the main result let us show some properties of the mapping  $\Phi_\mu$  restricted to  $\mathfrak{M}_L^0(\mathbb{R}^+)$ , which we denote by  $\Phi_\mu^0$ .

**Proposition 6** *Let  $\mu \in ID(\mathbb{R})$  with characteristic exponent  $\phi$ . Then  $\Phi_\mu^0$  has the following properties:*

1. *We have that  $\Phi_\mu^0(\mathfrak{M}_L^0(\mathbb{R}^+)) \subsetneq \mathfrak{M}_L^0(\mathbb{R}^+)$ .*
2. *If  $\nu = \Phi_\mu^0(\rho)$  for some  $\rho \in \mathfrak{M}_L^0(\mathbb{R}^+)$ , then any continuous and bounded complex-valued function is  $\nu$ -integrable and*

$$\int_{\mathbb{R}} e^{i\theta x} \nu(dx) = \int_0^\infty e^{s\phi(\theta)} \rho(ds), \quad \theta \in \mathbb{R}. \quad (14)$$

**Proof.** 1. Since for any  $\rho \in \mathfrak{M}_L^0(\mathbb{R}^+)$ ,  $\Phi_\mu^0(\rho)(\mathbb{R}) = \rho(\mathbb{R}^+) < \infty$ , we get that  $\Phi_\mu^0(\mathfrak{M}_L^0(\mathbb{R}^+)) \subset \mathfrak{M}_L^0(\mathbb{R}^+)$ . The strictly inclusion follows from Proposition 4.

2. Let  $\nu = \Phi_\mu^0(\rho)$  for some  $\rho \in \mathfrak{M}_L^0(\mathbb{R}^+)$ . From the previous point  $\Phi_\mu^0(\rho) \in \mathfrak{M}_L^0(\mathbb{R}^+)$ , thus for any real-valued continuous and bounded function  $f$  we have that

$$\int_{\mathbb{R}} |f(x)| \nu(dx) = \int_0^\infty \int_{\mathbb{R}} |f(x)| \mu^s(dx) \rho(ds) < \infty.$$

If  $f$  is complex-valued, continuous and bounded, its real part and its imaginary part are continuous and bounded as well, so integrable. Finally, since  $\mu^s$  is the probability measure with characteristic function  $e^{s\phi(\theta)}$ , (14) follows easily.  $\blacksquare$

**Remark 7** Observe that equation (14) only holds when  $\rho \in \mathfrak{M}_L^0(\mathbb{R}^+)$ . Indeed, put  $\rho(dx) = x^{-\alpha-1}dx$  for  $x > 0$  and  $0 < \alpha < 1$ . Then  $\rho \in \mathfrak{M}_L^1(\mathbb{R}^+)$  but  $\rho \notin \mathfrak{M}_L^0(\mathbb{R}^+)$  and for any  $z \in \mathbb{C}$  with  $\text{Re } z \leq 0$ ,  $\int_0^\infty e^{zs} \rho(ds)$  does not exist.

Next, we present the one-to-one property of  $\Phi_\mu^0$ . Recall that  $x_0 \in A \subset \mathbb{C}$  is a condensation point if for every  $\varepsilon > 0$ ,  $B_\varepsilon(x_0) \cap A \setminus \{x_0\} \neq \emptyset$ , where  $B_\varepsilon(x_0)$  is the open ball in  $\mathbb{C}$  with radius  $\varepsilon$  and center  $x_0$ .

**Theorem 8** *Let  $\mu \in ID(\mathbb{R})$  with characteristic triplet  $(\gamma, b, \nu)$ . Then, the mapping  $\Phi_\mu^0$  is one-to-one on its own domain.*

**Proof.** Let  $\rho, \tilde{\rho} \in \mathfrak{M}_L^0(\mathbb{R}^+)$  such that  $\Phi_\mu^0(\rho) = \Phi_\mu^0(\tilde{\rho})$ . From the proof of Proposition 6,  $\rho$  and  $\tilde{\rho}$  are finite measures with the same mass, so without loss of generality we may and do assume that they are probability measures. By equation (14)

$$\int_0^\infty e^{sz} \rho(ds) = \int_0^\infty e^{sz} \tilde{\rho}(ds), \quad \forall z \in \mathcal{A},$$

where

$$\mathcal{A} := \{\phi(\theta) : \theta \in \mathbb{R}\}. \quad (15)$$



We claim that there is  $\theta_0 \in \mathbb{R}$  for which  $\operatorname{Re}\phi(\theta_0) < 0$ . If this were true, due to the continuity of  $\phi$ ,  $\phi(\theta_0)$  would be a condensation point of  $\mathcal{A}$  having  $\operatorname{Re}\phi(\theta_0) < 0$  and since the set  $(-\infty, 0) \times i\mathbb{R}$  is contained in the interior of the domain of the Laplace transforms of  $\rho$  and  $\tilde{\rho}$ , this would imply that  $\rho$  and  $\tilde{\rho}$  coincides (see Chapter 4 in Hoffmann-Jørgensen (1994)), finishing thus the proof. Hence, we only need to show that such a  $\theta_0$  exists. Suppose the opposite, this is  $\operatorname{Re}\phi \geq 0$ . Since for any  $\theta \in \mathbb{R}$

$$\operatorname{Re}\phi(\theta) = -\frac{1}{2}b\theta^2 - \int_{\mathbb{R} \setminus \{0\}} [1 - \cos(\theta x)] \nu(dx) \leq 0,$$

we conclude that  $\operatorname{Re}\phi = 0$  which is equivalent to saying that  $b = 0$  and

$$\int_{\mathbb{R} \setminus \{0\}} [1 - \cos(\theta x)] \nu(dx) = 0, \quad \forall \theta \in \mathbb{R}.$$

Since  $1 - \cos(\theta x) \geq 0$ , from the previous equation we deduce that  $\cos(\theta \cdot) \equiv 1$   $\nu$ -a.e. and for all  $\theta \in \mathbb{R}$ , which is impossible. Thus, such a  $\theta_0$  must exist.  $\blacksquare$

**Corollary 9 (Probability measures)** *Let  $\mathcal{P}(\mathbb{R}^+)$  be the space of probability measures on  $\mathbb{R}^+$ . Then  $\Phi_\mu$  restricted to  $\mathcal{P}(\mathbb{R}^+)$  is one-to-one.*

### 3.3 The symmetric case

The situation of  $\mu$  being symmetric is of great interest, not only for its tractability but also because in this case  $\Phi_\mu$  is injective in its whole domain.

The following theorem is the main result of this subsection.

**Theorem 10** *Suppose that  $\mu \in ID(\mathbb{R})$  is symmetric. Then  $\Phi_\mu$  is one-to-one on its own domain.*

This result is basically a consequence of Corollary 9 and the following proposition:

**Proposition 11** *Assume that  $\mu \in ID(\mathbb{R})$  is symmetric and let  $\rho, \tilde{\rho} \in \mathfrak{M}_L^1(\mathbb{R}^+)$ . Then  $\Phi_\mu(\rho) = \Phi_\mu(\tilde{\rho})$  if and only if there exist (not necessarily unique)  $\eta_\rho, \eta_{\tilde{\rho}} \in ID(\mathbb{R}^+)$  having Lévy measures  $\rho$  and  $\tilde{\rho}$ , respectively, such that  $\Phi_\mu(\eta_\rho) = \Phi_\mu(\eta_{\tilde{\rho}})$ .*

**Proof.** Let us start by assuming that there are two infinitely divisible distributions  $\eta_\rho, \eta_{\tilde{\rho}} \in ID(\mathbb{R}^+)$  with Lévy measures  $\rho$  and  $\tilde{\rho}$ , such that  $\Phi_\mu(\eta_\rho) = \Phi_\mu(\eta_{\tilde{\rho}})$ . From Corollary 9, it follows immediately that  $\eta_\rho = \eta_{\tilde{\rho}}$ . Thus, by uniqueness of the characteristic triplet, it follows that  $\rho = \tilde{\rho}$ .

Conversely, suppose that  $\Phi_\mu(\rho) = \Phi_\mu(\tilde{\rho})$ . First note that in general  $\Phi_\mu(\rho) \equiv 0$  if and only if  $\rho \equiv 0$ , so without loss of generality, we may and do assume that  $\rho, \tilde{\rho} \neq 0$ . Let,  $T^\rho$  and  $T^{\tilde{\rho}}$  be two purely discontinuous subordinators with Lévy measures  $\rho$  and  $\tilde{\rho}$ , respectively. Put  $\eta_\rho$  and  $\eta_{\tilde{\rho}}$  to be the distributions related to  $T^\rho$  and  $T^{\tilde{\rho}}$  at time 1. Let  $\bar{\mu} = \mathcal{L}(L_{T_1^\rho})$  and  $\bar{\mu}' = \mathcal{L}(L_{T_1^{\tilde{\rho}}})$ , where  $L$  is the Lévy process associated to  $\mu$ . Since  $\bar{\mu} = \Phi_\mu(\eta_\rho)$  and  $\bar{\mu}' = \Phi_\mu(\eta_{\tilde{\rho}})$ , then  $\Phi_\mu(\eta_\rho) = \Phi_\mu(\eta_{\tilde{\rho}})$  if and only if the characteristic triplets of  $\bar{\mu}$  and  $\bar{\mu}'$  coincide. From Theorem 1 we get that  $\bar{\mu}$  and  $\bar{\mu}'$  have characteristic triplets  $(\gamma, 0, \nu)$  and  $(\gamma', 0, \nu')$  given by  $\nu = \Phi_\mu(\rho)$ ,  $\nu' = \Phi_\mu(\tilde{\rho})$  and

$$\gamma = \int_0^\infty \int_{|x| \leq 1} x \mu^s(dx) \rho(ds); \quad \gamma' = \int_0^\infty \int_{|x| \leq 1} x \mu^s(dx) \tilde{\rho}(ds).$$

By hypothesis  $\nu = \tilde{\nu}$ . Moreover, since  $\mu$  is symmetric  $\int_{|x| \leq 1} x \mu^s(dx) = 0$  for any  $s > 0$ , it follows that  $\gamma = \gamma' = 0$ , which means that  $\bar{\mu} = \bar{\mu}'$ , as required. To finish the proof, note that  $\eta_\rho$  and  $\eta_{\tilde{\rho}}$  are not unique. To see this, it is enough to consider  $T^\rho$  and  $T^{\tilde{\rho}}$  to be two subordinators with the same drift and Lévy measures  $\rho$  and  $\tilde{\rho}$ . In this case, by following the same reasoning as above, we get that  $\Phi_\mu(\eta_\rho) = \Phi_\mu(\eta_{\tilde{\rho}})$ , where, as before,  $\eta_\rho$  and  $\eta_{\tilde{\rho}}$  are the laws of  $T^\rho$  and  $T^{\tilde{\rho}}$ , respectively.  $\blacksquare$

Now we present a proof of Theorem 10:

**Proof of Theorem 10.** Assume that  $\mu$  is symmetric. Let  $\rho, \tilde{\rho} \in \mathfrak{M}_L^1(\mathbb{R}^+)$  such that  $\Phi_\mu(\rho) = \Phi_\mu(\tilde{\rho})$ . From the previous proposition there exist  $\eta_\rho, \eta_{\tilde{\rho}} \in ID(\mathbb{R}^+)$  having Lévy measures  $\rho$  and  $\tilde{\rho}$ , respectively, such that  $\Phi_\mu(\eta_\rho) = \Phi_\mu(\eta_{\tilde{\rho}})$ . An application of Corollary 9 results in  $\eta_\rho = \eta_{\tilde{\rho}}$ , which obviously implies that  $\rho = \tilde{\rho}$ . ■

**Remark 12** Due to Proposition 11 and Theorem 10 we have that the injectivity of  $\Phi_\mu$  is equivalent to the one of  $\Phi_\mu^0$  in the symmetric case.

Recall that the Upsilon transformation of  $\eta$  via  $\rho$  is defined as the  $\sigma$ -finite measure given by  $\Upsilon_\rho(\eta)(\{0\}) = 0$  and

$$\Upsilon_\rho(\eta)(A) = \int_0^\infty \eta(s^{-1}A) \rho(ds), \quad A \in \mathcal{B}(\mathbb{R} \setminus \{0\}).$$

**Example 13** Suppose that  $\mu \in ID(\mathbb{R})$  is a symmetric strictly  $\alpha$ -stable distribution with  $0 < \alpha \leq 2$ . By the strict stability, we have that for all  $s > 0$

$$\mu^s(dx) = \mu(s^{-1/\alpha}dx).$$

Thus, if  $\rho \in \mathfrak{M}_L^1(\mathbb{R}^+)$

$$\begin{aligned} \Phi_\mu(\rho)(dx) &= \int_0^\infty \mu(s^{-1/\alpha}dx) \rho(ds) \\ &= \int_0^\infty \mu(s^{-1}dx) \rho_\alpha(ds), \end{aligned}$$

where  $\rho_\alpha$  is the measure induced by  $\rho$  and the mapping  $s \mapsto s^{1/\alpha}$ . Therefore  $\Phi_\mu(\rho)$  equals  $\Upsilon_{\rho_\alpha}(\mu)$ , the upsilon transformation of  $\mu$  via  $\rho_\alpha$ . In particular, when  $\mu$  is concentrated in  $(0, \infty)$

$$\Phi_\mu(\rho) = \Upsilon_\mu(\rho_\alpha) = \mu \circledast \rho_\alpha, \quad \rho \in \mathfrak{M}_L^1(\mathbb{R}^+). \quad (16)$$

where  $\circledast$  is the multiplicative convolution of measures, this is

$$\nu \circledast \eta(A) = \int_0^\infty \int_0^\infty 1_A(xs) \eta(dx) \nu(ds), \quad A \in \mathcal{B}(\mathbb{R}^+),$$

for two  $\sigma$ -finite measures on  $\mathcal{B}(\mathbb{R}^+)$ . In this context, the injectivity of  $\Phi_\mu$  is equivalent to the injectivity of  $\Upsilon_\mu$  and the *cancellation property* of  $\mu$  (see for instance Jacobsen et al. (2009) and Pedersen and Sauri (2014)). In the next subsection we study these problems.

### 3.4 The case of $\mu$ supported on $(0, \infty)$

In the symmetric case, the key point for the injectivity of  $\Phi_\mu$  is Proposition 11. In this subsection we will show that such result also holds in the case when  $\mu$  is concentrated on  $(0, \infty)$ .

**Proposition 14** Suppose that  $\mu \in ID(\mathbb{R})$  has support on  $(0, \infty)$ . Then the statement of Proposition 11 holds.

**Proof.** The argument in the proof of Proposition 11 remains the same in this case, except that  $\int_{|x| \leq 1} x \mu^s(dx) = 0$  for any  $s > 0$ , which of course is not true when  $\mu$  has support on  $(0, \infty)$ . Therefore, under the notation of the proof of Proposition 11, we only need to show that  $\gamma = \gamma'$ . Since  $\mu$  has support on  $(0, \infty)$ , it corresponds to the law of a subordinator. Consequently,  $\overline{\mu}$  and  $\overline{\mu}'$  are also

concentrated on  $(0, \infty)$  and for any non-negative measurable function  $f$  vanishing on a neighborhood of zero we have that

$$\int_{\mathbb{R}} f(x) \nu(dx) = \int_0^\infty \int_0^\infty f(x) \mu^s(dx) \rho(ds),$$

and

$$\int_{\mathbb{R}} f(x) \nu'(dx) = \int_0^\infty \int_0^\infty f(x) \mu^s(dx) \tilde{\rho}(ds).$$

Thus, by the Monotone Convergence Theorem

$$\gamma = \lim_{\varepsilon \downarrow 0} \int_0^\infty \int_\varepsilon^1 x \mu^s(dx) \rho(ds) = \lim_{\varepsilon \downarrow 0} \int_\varepsilon^1 x \nu(dx),$$

and

$$\gamma' = \lim_{\varepsilon \downarrow 0} \int_0^\infty \int_\varepsilon^1 x \mu^s(dx) \tilde{\rho}(ds) = \lim_{\varepsilon \downarrow 0} \int_\varepsilon^1 x \nu'(dx),$$

but by hypothesis  $\nu = \nu'$ , consequently  $\gamma = \gamma'$ , as required.  $\blacksquare$

By the previous proposition and Remark 12, the invertibility of  $\Phi_\mu$  is equivalent to the injectivity of  $\Phi_\mu^0$ , or in other words:

**Theorem 15** *Let  $\mu \in ID(\mathbb{R})$  having support on  $(0, \infty)$ . Then,  $\Phi_\mu$  is one-to-one on its own domain and  $\Phi_\mu(\mathfrak{M}_L^1(\mathbb{R}^+)) \subsetneq \mathfrak{M}_L^1(\mathbb{R}^+)$ .*

**Example 16 (Gamma process)** Suppose that

$$\mu(dx) = \lambda e^{-\lambda x} \mathbf{1}_{(0, \infty)}(x) dx, \quad \lambda > 0. \quad (17)$$

From the previous theorem, the induced mapping  $\Phi_\mu$  is one-to-one. However, a direct proof can be given: We know that in this case, for any  $s > 0$

$$\mu^s(dx) = \frac{(\lambda x)^s}{x \Gamma(s)} e^{-\lambda x} \mathbf{1}_{(0, \infty)}(x) dx.$$

Thus,  $\Phi_\mu(\rho)$  is absolutely continuous with density given by  $f_\rho(0) = 0$  and for  $x > 0$

$$f_\rho(x) = \frac{e^{-\lambda x}}{x} \int_0^\infty \frac{(\lambda x)^s}{\Gamma(s)} \rho(ds). \quad (18)$$

We claim that the transformation  $\rho \mapsto f_\rho$  is uniquely determined by  $\rho$ . Let  $\tilde{\rho} \in \mathfrak{M}_L^1(\mathbb{R}^+)$ , such that  $f_\rho = f_{\tilde{\rho}}$ . Then,  $f_\rho(0) = f_{\tilde{\rho}}(0)$  and from (18), for all  $x > 0$

$$\int_0^\infty \frac{(\lambda x)^s}{\Gamma(s)} \rho(ds) = \int_0^\infty \frac{(\lambda x)^s}{\Gamma(s)} \tilde{\rho}(ds). \quad (19)$$

Consider the measures

$$\rho_\Gamma(ds) = \frac{1}{\Gamma(s)} \rho(ds); \quad \tilde{\rho}_\Gamma(ds) = \frac{1}{\Gamma(s)} \tilde{\rho}(ds).$$

Since the gamma function behaves as  $s^{-1}$  near zero, we have that  $\rho_\Gamma$  and  $\tilde{\rho}_\Gamma$  are finite measures. Furthermore, from (19),  $\rho_\Gamma$  and  $\tilde{\rho}_\Gamma$  have the same mass, so without loss of generality we may and do assume that  $\rho_\Gamma$  and  $\tilde{\rho}_\Gamma$  are probability measures satisfying

$$\int_0^\infty x^s \rho_\Gamma(ds) = \int_0^\infty x^s \tilde{\rho}_\Gamma(ds), \quad \text{for all } x > 0.$$

Thus, by the uniqueness of the generating function (see Chapter 4 in Hoffmann-Jørgensen (1994)), we conclude that  $\rho_\Gamma = \tilde{\rho}_\Gamma$  or  $\rho = \tilde{\rho}$ , because the gamma function is continuous and strictly positive in  $(0, \infty)$ .

**Example 17 ( $\alpha$ -stable distribution)** Suppose that  $\mu \in ID(\mathbb{R}^+)$  is a strictly  $\alpha$ -stable distribution for  $0 < \alpha < 1$ . From equation (16) in Example 13, the mapping  $\Phi_\mu$  corresponds to the upilon transformation

$$\Upsilon_\mu(\eta)(A) = \int_0^\infty \int_0^\infty 1_A(xs) \eta(ds) \mu(dx), \quad A \in \mathcal{B}(\mathbb{R} \setminus \{0\}).$$

According to Theorem 3.4 in Barndorff-Nielsen et al. (2008), the Lévy domain of  $\Upsilon_\mu$  cannot be neither  $\mathfrak{M}_L(\mathbb{R}^+)$  nor  $\mathfrak{M}_L^1(\mathbb{R}^+)$  because  $\mu$  has moments strictly smaller than  $\alpha$ . Nevertheless, by equation (16),  $\Upsilon_\mu(\eta) \in \mathfrak{M}_L(\mathbb{R})$  if the image measure of  $\eta$  and the mapping  $s \mapsto s^\alpha$  belong to  $\mathfrak{M}_L^1(\mathbb{R}^+)$ . In such set of measures,  $\Upsilon_\mu$  is one-to-one. The cancellation property of  $\mu$  is obtained in an analogous way.

## 4 On the law of a subordinated Lévy process

In this section we study the law associated with a subordinated Lévy process and we show that this class is in a bijection with the infinitely divisible distributions associated with subordinators. For a fixed Lévy process, as we will see below, the law of a subordinated Lévy process appears as the restriction of the mapping  $\Phi_\mu$  to the space of infinitely divisible distributions with support on  $(0, \infty)$ . In what follows,  $\mathcal{L}(X)$  will denote the distribution of the random variable  $X$ .

Let us start by introducing the mappings which we are interested in. Let  $\mu_L, \mu_T \in ID(\mathbb{R})$  with characteristic triplets  $(\gamma, b, \nu)$  and  $(\beta_0, 0, \rho)_0$ , respectively. Put  $\bar{\mu} \in ID(\mathbb{R})$  as the probability distribution with characteristic triplet (relative to  $\tau$ )  $(\bar{\gamma}, \bar{b}, \bar{\nu})$  given by

$$\begin{aligned} \bar{b} &= b\beta_0; \\ \bar{\gamma} &= \gamma\beta_0 + \int_0^\infty \int_{|x| \leq 1} x\mu^s(dx) \rho(ds); \\ \bar{\nu}(dx) &= \beta_0\nu(dx) + \int_0^\infty \mu^s(dx) \rho(ds). \end{aligned} \tag{20}$$

Define  $\Lambda_{\mu_L} : \mathcal{D}(\Lambda_{\mu_L}) \rightarrow ID(\mathbb{R})$ ,  $\Lambda_{\mu_T} : \mathcal{D}(\Lambda_{\mu_T}) \rightarrow ID(\mathbb{R})$  and  $\Lambda : \mathcal{D}(\Lambda) \rightarrow ID(\mathbb{R})$  by the formulae

$$\begin{aligned} \Lambda_{\mu_L}(\mu_T) &= \bar{\mu}; \\ \Lambda_{\mu_T}(\mu_L) &= \bar{\mu}; \\ \Lambda(\mu_L, \mu_T) &= \bar{\mu}, \end{aligned}$$

for some domains  $\mathcal{D}(\Lambda_{\mu_L})$ ,  $\mathcal{D}(\Lambda_{\mu_T})$  and  $\mathcal{D}(\Lambda)$ . Observe that  $\Lambda_{\mu_L}$  and  $\Lambda_{\mu_T}$  are the sections of  $\Lambda$  for fixed  $\mu_L$  and  $\mu_T$ , respectively.

In the following remarks  $\bar{\mu}$  is the probability measure associated with  $(\bar{\gamma}, \bar{b}, \bar{\nu})$  as in (20).

**Remark 18** Observe that in general,  $(\bar{\gamma}, \bar{b}, \bar{\nu})$  is the characteristic triplet of an infinitely divisible distribution if and only if  $\beta_0 \geq 0$  and  $\rho \in \mathcal{D}_L(\Phi_{\mu_L})$ .

**Remark 19** Let  $L$  be a Lévy process and  $T$  a subordinator with characteristic triplet  $(\gamma, b, \nu)$  and pair  $(\beta_0, \rho)_0$ , respectively. Consider  $L_T$  to be the Lévy process obtained by subordinating  $L$  through  $T$ . Then, by Theorem 1 and (13)

$$\bar{\mu} = \mathcal{L}(L_{T_1}) = \Phi_{\mu_L}(\mu_T), \quad \mu_T \sim T_1, \mu_L \sim L_1. \tag{21}$$

**Remark 20** Observe that  $\Lambda$  is not one-to-one in general, for instance if  $\mu_T \sim (a, 0, 0)$ ,  $\mu_L \sim (0, b, 0)$ ,  $\tilde{\mu}_T \sim (a^2, 0, 0)$  and  $\tilde{\mu}_L \sim (0, \frac{1}{a}b, 0)$ . Then, by (20)

$$\Lambda(\mu_L, \mu_T) = \Lambda(\tilde{\mu}_L, \tilde{\mu}_T).$$

For the rest of this section, we study some properties of  $\Lambda_{\mu_L}$  and  $\Lambda_{\mu_T}$ , including their injectivity. Let  $\mathcal{R}(\Lambda_{\mu_L})$  and  $\mathcal{R}(\Lambda_{\mu_T})$  be the range of  $\Lambda_{\mu_L}$  and  $\Lambda_{\mu_T}$ , respectively. Denote by  $ID(\mathbb{R}^+)$  the subset of infinitely divisible distributions with support on  $\mathbb{R}^+$ .

**Proposition 21** *We have that  $\mathcal{D}(\Lambda_{\mu_L}) = ID(\mathbb{R}^+)$  and  $\mathcal{D}(\Lambda_{\mu_T}) = ID(\mathbb{R})$  and in general the ranges  $\mathcal{R}(\Lambda_{\mu_L})$  and  $\mathcal{R}(\Lambda_{\mu_T})$  are proper subsets of  $ID(\mathbb{R})$ . Furthermore,  $\mathcal{R}(\Lambda_{\mu_L})$  is closed under convolutions and*

$$\Lambda_{\mu_L}(\mu^1 * \mu^2) = \Lambda_{\mu_L}(\mu^1) * \Lambda_{\mu_L}(\mu^2), \quad \mu^1, \mu^2 \in \mathcal{D}(\Lambda_{\mu_L}). \quad (22)$$

*In contrast,  $\mathcal{R}(\Lambda_{\mu_T})$  is not closed under convolutions in general.*

**Proof.** The first part is derived directly from the definition of  $\Lambda_{\mu_L}$  and  $\Lambda_{\mu_T}$ , Proposition 4 and Remark 18. Now, we proceed to verify (22). Let  $\mu_L \in ID(\mathbb{R})$ . Suppose that  $\mu \in \mathcal{D}(\Lambda_{\mu_L})$  and has pair  $(\beta_0, \rho)_0$ . Then by Remark 19 and Theorem 30.1 in Sato (1999)

$$\log \widehat{\Lambda_1}(\mu)(\theta) = \psi_\mu(\log \widehat{\mu_L}(\theta)), \quad \theta \in \mathbb{R}, \quad (23)$$

where

$$\psi_\mu(z) := \log \int_0^\infty e^{zx} \mu(dx) < \infty, \quad z \in \mathbb{C}, \operatorname{Re}(z) \leq 0. \quad (24)$$

Hence, if  $\mu^1, \mu^2 \in \mathcal{D}(\Lambda_{\mu_L})$ , using that  $\psi_{\mu^1 * \mu^2} = \psi_{\mu^1} + \psi_{\mu^2}$ , we get that for any  $\theta \in \mathbb{R}$

$$\begin{aligned} \log \widehat{\Lambda_{\mu_L}}(\mu^1 * \mu^2)(\theta) &= \psi_{\mu^1 * \mu^2}(\log \widehat{\mu_L}(\theta)) \\ &= \psi_{\mu^1}(\log \widehat{\mu_L}(\theta)) + \psi_{\mu^2}(\log \widehat{\mu_L}(\theta)) \\ &= \log \widehat{\mu^1 * \mu^2}(\theta), \end{aligned}$$

where  $\Lambda_{\mu_L}(\mu^1) = \widehat{\mu^1}$  and  $\Lambda_{\mu_L}(\mu^2) = \widehat{\mu^2}$ , and  $\widehat{\eta}$  stands for Fourier transform of the measure  $\eta$ . This shows (22) and in particular it follows that  $\mathcal{R}(\Lambda_{\mu_L})$  is closed under convolutions. The last part is obtained by noting that in general for a fix  $\mu_T \in ID(\mathbb{R}^+)$

$$\psi_{\mu_T}(\log \widehat{\mu^1}(\theta) + \log \widehat{\mu^2}(\theta)) \neq \psi_{\mu_T}(\log \widehat{\mu^1}(\theta)) + \psi_{\mu_T}(\log \widehat{\mu^2}(\theta)), \quad \theta \in \mathbb{R}.$$

■

Note that in what follows, by *continuity* we mean continuity with respect to the weak convergence of probability measures.

**Proposition 22** *The mappings  $\Lambda_{\mu_L}$  and  $\Lambda_{\mu_T}$  are continuous.*

**Proof.** Suppose that  $\mu_L \in ID(\mathbb{R})$  and let  $\mu \in \mathcal{D}(\Lambda_{\mu_L})$ . Then  $\Lambda_{\mu_L}(\mu)$  has characteristic exponent given by (23). Let  $(\mu_n)_{n \geq 1} \subset \mathcal{D}(\Lambda_{\mu_L})$ , such that  $\mu_n \rightarrow \mu_\infty$  weakly. From the previous proposition  $\mu_\infty \in \mathcal{D}(\Lambda_{\mu_L})$ . Thus, due to the Continuity Theorem,  $\psi_{\mu_n} \rightarrow \psi_{\mu_\infty}$  pointwise, where  $\psi_\mu$  is as in (24). The continuity of  $\Lambda_{\mu_L}$  follows by applying the previous observations to (23). Finally, by interchanging the roles in the previous reasoning, it follows that  $\Lambda_{\mu_T}$  is continuous as well. ■

From Propositions 21 and 22, for a given  $\mu_L \in ID(\mathbb{R})$ ,  $\Lambda_{\mu_L}$  is continuous and linear with respect to the convolution operation. Moreover, Corollary 9 and (21) show that  $\Lambda_{\mu_L}$  is one-to-one. Hence, we have obtained the following result:

**Theorem 23** *Let  $\mu_L \in ID(\mathbb{R})$  be given. If  $\mu_L$  is not the Dirac's delta measure at zero, then  $\Lambda_{\mu_L}$  is a continuous group isomorphism. Moreover,  $\Lambda_{\mu_L}(ID(\mathbb{R})) \subsetneq ID(\mathbb{R})$ .*

To conclude this part we present the injectivity of  $\Lambda_{\mu_T}$ .

**Theorem 24** *Let  $\mu_T \in ID(\mathbb{R}^+)$  be given. If  $\mu_T$  is not the Dirac delta measure at zero, then  $\Lambda_{\mu_T}$  is one-to-one. Moreover,  $\Lambda_{\mu_T}(ID(\mathbb{R})) \subsetneq ID(\mathbb{R})$ .*

**Proof.** Assume that  $\mu_T \in ID(\mathbb{R}^+)$  is not the Dirac delta measure at zero and let  $\mu, \tilde{\mu} \in ID(\mathbb{R})$  such that  $\Lambda_{\mu_T}(\mu) = \Lambda_{\mu_T}(\tilde{\mu})$ . Define by  $L, \tilde{L}$  and  $T$  the Lévy processes associated with  $\mu, \tilde{\mu}$  and  $\mu_T$ , respectively. Since  $\mu_T \in ID(\mathbb{R}^+)$  and is not the Dirac delta measure at zero,  $T$  is a non-zero subordinator. Suppose that the Lévy measure of  $\mu_T$  is infinity in a neighborhood of zero, then  $T_t > 0$  almost surely for any  $t > 0$ . Consequently, by independence, for all  $t > 0$

$$\begin{aligned}\mathbb{E}(e^{i\theta L_{T_1}} | T) &= e^{T_1 \log \hat{\mu}}; \\ \mathbb{E}(e^{i\theta \tilde{L}_{T_1}} | T) &= e^{T_1 \log \hat{\tilde{\mu}}}.\end{aligned}$$

But since  $\Lambda_{\mu_T}(\mu) = \Lambda_{\mu_T}(\tilde{\mu})$ , we have that almost surely

$$e^{T_1 \log \hat{\mu}} = e^{T_1 \log \hat{\tilde{\mu}}}. \quad (25)$$

From this, it is immediate that  $\mu = \tilde{\mu}$ . If the Lévy measure of  $\mu_T$  is finite, then the first jump time of  $T$  is finite almost surely. Following the same reasoning as before, we get that (25) holds for  $T_{e_1}$  instead of  $T_1$ , where  $e_1$  is the first jump time of  $T$ . This concludes the proof. ■

## 5 On the recovery problem of time-changed infinitely divisible processes

In this section we study the recovery problem of subordinated Lévy bases and  $\mathcal{LSS}$  processes driven by a subordinated Lévy process.

Let  $(L_t)_{t \geq 0}$  and  $(T_t)_{t \geq 0}$  be a Lévy process and an increasing process starting at zero and independent of  $L$  with càdlàg paths, respectively. Consider  $(X_t)_{t \geq 0}$  given by

$$X_t := L_{T_t}, \quad t \geq 0, \quad (26)$$

which is the process obtained by time changing  $L$  via  $T$ . These kind of processes are of importance in mathematical finance since they are often used to model the return process of a financial asset. In this context, when  $L$  is a Brownian motion,  $T$  plays the role of *stochastic volatility*. Since in reality  $T$  cannot be observed, a very important question arises: Is there a way to *recover*  $T$  from  $X$ ? More generally, is  $T$  *completely determined* by  $X$ , either in the *path-wise sense* or in the *distributional sense*? This issue is what we refer to as the *recovery problem* of time-changed Lévy processes.

Observe that when  $L$  is a standard Brownian motion and  $T$  is continuous the recovery problem is trivial, because in this case  $X$  is continuous and

$$T_t = [X]_t, \quad \text{a.s. for all } t > 0.$$

Winkel (2001) studied the problem when  $L$  is a Lévy process (but not compound Poisson) and  $T$  being continuous. In this case  $T$  can be expressed in terms of the Lévy measure of  $L$ , the quadratic variation of  $X$  as well as its jumps.

Geman et al. (2002) studied the conditional law of  $X$  when  $L$  is a Brownian motion and  $T$  is a subordinator with pair  $(\beta_0, \rho)_0$ . They showed that

$$\mathbb{E}(e^{-\lambda T_t} | \mathcal{F}_t) = e^{-t\psi(\lambda)} M_t(\lambda, \rho, X), \quad \lambda, t > 0,$$

where  $\psi$  is the characteristic exponent of the Laplace transform of  $T$ ,  $(\mathcal{F}_t)_{t \geq 0}$  is the natural filtration of  $X$  and  $M_t(\lambda, \rho, X)$  is certain martingale depending only on  $\lambda, \rho$  and  $(\bar{X}_s)_{0 \leq s \leq t}$ . Winkel (2001) improved this result to include any Lévy process for which its associated law has density. In this case

$$\mathbb{E}(e^{-\lambda T_t} | \mathcal{F}_t) = e^{-t\psi(\lambda)} M_t(\lambda, \rho, X, \mu), \quad \lambda, t > 0,$$

with  $M_t(\lambda, \rho, X, \mu)$  certain martingale depending only on  $\lambda, \rho, (X_s)_{0 \leq s \leq t}$  and  $\mu \in ID(\mathbb{R})$ , the law of  $L$  at time 1. Furthermore, the author shows that if  $T$  is purely discontinuous, then  $X$  and  $T$  jumps at the same time, and

$$\mathbb{P}(\Delta T_t \in dz | S = t, \Delta X_t = y) = \frac{1}{c(\mu, \rho)} f(y, z) \rho(dz),$$

where  $c(\mu, \rho)$  is a positive constant depending on the Lévy measure of  $T$  and the law of  $L$ .

Nevertheless, in all the cases above, *it is implicitly assumed that the triplet of  $T$  can be obtained by  $X$* . What we have shown in the previous section is that the triplet of  $T$  is uniquely determined by  $X$  for every non-zero Lévy process.

For the rest of the section, we study the distributional recovery problem for subordinated Lévy bases and for the class of Lévy semistationary processes.

### 5.1 The recovery problem of subordinated Lévy bases

In this part we study the recovery problem for subordinated Lévy bases. It turned out that the non-homogeneous case is deduced directly from the homogeneous case.

Recall that any arbitrary Lévy basis  $T$  on  $\mathbb{R}^k$  with characteristic quadruplet  $(\beta(s), b(s), \nu(s, \cdot), c(s))$  is a measure-valued field if and only if for  $c$ -almost all  $s$ ,  $b(s) = 0$ ,  $\nu(s, (-\infty, 0)) = 0$ , the integral  $\int_{\mathbb{R}^+} (1 \wedge x) \nu(s, dx)$  is finite and  $\beta(s) \geq \int_0^1 x \nu(s, dx)$  (see Section 2). This is equivalent to saying that the laws of the Lévy seeds  $\mathbb{T} = (T'(s))_{s \in \mathbb{R}^k}$ , say,  $M = (\mu_s)_{s \in \mathbb{R}^k}$ , have support on  $[0, \infty)$ . Let  $L$  be a homogeneous Lévy basis with characteristic triplet  $(\gamma, b, \nu)$  and  $T$  as before. Put  $L_T$  to be as in (11), i.e.  $L_T$  is the Lévy basis subordinated by the meta-time associated with  $T$ . Within this framework,  $T$  is uniquely determined by  $L_T$  in law, as the following theorem shows:

**Theorem 25** *Under the notation above, if  $\mu \in ID(\mathbb{R})$  has characteristic triplet  $(\gamma, b, \nu)$ , and it is not the Dirac delta measure at zero, then the law of  $L_T$  is determined uniquely by the law of  $T$ , i.e. if there exists a non-negative Lévy basis  $\tilde{T}$ , such that  $L_T \stackrel{d}{=} L_{\tilde{T}}$ , then necessarily  $T \stackrel{d}{=} \tilde{T}$ .*

**Proof.** Let  $T$  and  $\tilde{T}$  be two non-negative Lévy bases and  $(\beta(s), 0, \nu(s, \cdot), c(s)), (\tilde{\beta}(s), 0, \tilde{\nu}(s, \cdot), \tilde{c}(s))$ , their corresponding characteristic quadruplets. Denote by  $M = (\mu_s)_{s \in \mathbb{R}^k}$  the collection of infinitely divisible laws related to the Lévy seeds of  $T$ , this is  $\mu_s = \mathcal{L}(T'(s))$ . Define  $\tilde{M} = (\tilde{\mu}_s)_{s \in \mathbb{R}^k}$  in an analogous way. From the observation made at the beginning of this subsection, we get that  $M, \tilde{M} \subset ID(\mathbb{R}^+)$ . Now, suppose that  $L_T \stackrel{d}{=} L_{\tilde{T}}$ . Due to Theorem 1, it follows that  $c = \tilde{c}$ . Hence,  $T \stackrel{d}{=} \tilde{T}$  if and only if  $\mu_s = \tilde{\mu}_s$  for  $c$ -almost all  $s \in \mathbb{R}^k$ . Let us show that this holds whenever  $L_T \stackrel{d}{=} L_{\tilde{T}}$ .

Since  $L_T$  is a Lévy basis, there is an associated collection of infinitely divisible laws  $(\bar{\mu}_s)_{s \in \mathbb{R}^k}$ , say, on  $\mathbb{R}$ , such that  $\bar{\mu}_s = \mathcal{L}(L'_T(s))$ , where  $L'_T(s)$  is the infinitely divisible random variable with characteristic triplet  $(\bar{\gamma}(s), \bar{b}(s), \bar{\nu}(s, dx))$ , with  $\bar{\gamma}(s), \bar{b}(s)$  and  $\bar{\nu}(s, dx)$  as in Theorem 1. An analogous expression holds for  $L_{\tilde{T}}$  and its associated Lévy seeds  $L'_{\tilde{T}}(s)$ . By hypothesis  $L'_T(s) \stackrel{d}{=} L'_{\tilde{T}}(s)$  for  $c$ -almost all  $s \in \mathbb{R}^k$ . Moreover, from (20) and (21), for all  $s \in \mathbb{R}^k$

$$\mathcal{L}(L'_T(s)) = \Lambda_\mu(\mu_s) = \Phi_\mu(\mu_s),$$

and

$$\mathcal{L}(L'_{\tilde{T}}(s)) = \Lambda_\mu(\tilde{\mu}_s) = \Phi_\mu(\tilde{\mu}_s).$$

Therefore, for  $c$ -almost all  $s \in \mathbb{R}^k$

$$\Lambda_\mu(\mu_s) = \Lambda_\mu(\tilde{\mu}_s),$$

which, once one applies Theorem 23, implies that  $\mu_s = \tilde{\mu}_s$  for  $c$ -almost all  $s \in \mathbb{R}^k$ , just as it was claimed.  $\blacksquare$

Due to the bijection between natural additive processes and Lévy bases on  $\mathbb{R}^+$ , the previous theorem implies the following:

**Corollary 26** *Let  $(L_t)_{t \geq 0}$  be a Lévy process and  $(T_t)_{t \geq 0}$  an increasing additive process. Then the law of  $(T_t)_{t \geq 0}$  is completely determined by the law of the time-changed process  $(X_t)_{t \geq 0}$ , where  $X_t$  is as in (26).*

**Proof.** It is enough to point out that in this case, by Theorem 4.2 in Pedersen (2003) and Example 3.3 in Barndorff-Nielsen and Pedersen (2012), there exist a homogeneous Lévy basis  $\mathbf{L}$  and a non-negative Lévy basis  $\mathbf{T}$ , such that

$$X_t = \mathbf{L}_{\mathbf{T}}([0, t]), \quad \text{a.s. for all } t > 0,$$

where  $\mathbf{L}_{\mathbf{T}}$  is the time-changed Lévy basis via  $\mathbf{T}$ . Thus, the result follows from the previous theorem.  $\blacksquare$

**Remark 27** Note that by reasoning as in Theorem 25, if  $(T_t)_{t \geq 0}$  is an arbitrary infinitely divisible chronometer (not necessarily with independent increments), we have that for all  $t > 0$

$$\mathcal{L}(X_t) = \Lambda_{\mu}(\mu_t),$$

where  $\mu_t$  is the law of  $T$  at time  $t$ . Hence, if there exists another infinitely divisible chronometer  $(\tilde{T}_t)_{t \geq 0}$  such that  $(L_{T_t})_{t \geq 0} \stackrel{d}{=} (L_{\tilde{T}_t})_{t \geq 0}$ , then necessarily for every  $t > 0$

$$T_t \stackrel{d}{=} \tilde{T}_t,$$

which in general does not imply that  $T \stackrel{d}{=} \tilde{T}$ . Therefore, more general mappings than  $\Phi_{\mu}$  should be considered, namely transformations on  $ID(\mathbb{R}^{(0, \infty)})$ , the space of infinitely divisible laws on  $\mathbb{R}^{(0, \infty)}$ , into itself. See Barndorff-Nielsen et al. (2014) for more details. We leave such a problem for future research.

## 5.2 Lévy semistationary processes and the recovery problem

Consider the subclass of Lévy semistationary processes given by the formula,

$$Y_t := \int_{-\infty}^t f(t-s) \sigma_s dL_s^T, \quad t \in \mathbb{R}. \quad (27)$$

where  $f$  is a deterministic function such that  $f(x) = 0$  for  $x \leq 0$ ,  $\sigma$  a càdlàg predictable process and  $X^T$  is a two-sided Lévy process for which

$$L_s^T := L_{T_s}, \quad s \geq 0,$$

with  $L$  a Lévy process with triplet  $(\gamma, b, \nu)$  and  $T$  a subordinator with couple  $(\beta_0, \rho)_0$ . Note that the process  $\sigma$  introduces stochastic volatility through *stochastic amplitude modulation* whereas  $T$  introduces stochastic volatility through *stochastic intensity modulation*. Thus,  $\sigma$  changes the height of the jumps and variation of  $L$  while  $T$  randomizes the jump times. Observe that  $Y$  also can be written as

$$Y_t = \int_{-\infty}^t f(t-s) dX_s, \quad t \in \mathbb{R},$$

where  $X$  is the increment semimartingale given by

$$X_t - X_s = \int_s^t \sigma_r dL_r^T, \quad s < t.$$

Therefore, in this case we may think in terms of two different kinds of recovery problems. The first, involving  $\sigma$  and the second related to  $T$ . If  $T_t = t$  and  $L$  is a Brownian motion then  $\sigma$  can be recovered



pathwise. See Theorem 3.1 in Corcuera et al. (2013). More generally, if  $Y$  is *invertible*, i.e.  $X_t$  can be obtained by limit of linear combinations of  $Y$ , hence  $\sigma$  can be recovered pathwise, because

$$[X]_t = \int_0^t \sigma_s^2 d[L]_s^T, \quad \text{a.s. } t \geq 0.$$

Note that in this general case, the law of  $Y$  is no longer infinitely divisible. Moreover, it is not clear whether  $T$  can be identified jointly with  $\sigma$ .

Let us consider some examples where the invertibility of  $Y$  plays an important role.

**Example 28 (Ornstein-Uhlenbeck processes)** Put  $f(x) = e^{-x}$  for  $x \geq 0$  and  $\sigma \equiv 1$ . Then  $Y$  is an Ornstein-Uhlenbeck process satisfying the Langevin equation

$$dY_t = Y_t dt + dL_t^T, \quad t \geq 0.$$

This implies that the law of  $Y$  is uniquely determined by the law of  $L^T$ , but since the law of  $L^T$  is uniquely determined by the law of  $T$  (for a fixed  $L$ ), we conclude that the law of  $Y$  is uniquely determined by the one of  $T$ . Here we can recover (in the distributional sense)  $T$  via  $Y$ .

**Example 29 (Semimartingale case)** It is well known that the process  $Y$  is not in general a semimartingale. Suppose that  $L$  is a square integrable martingale and  $Y$  is a semimartingale. Then  $Y$  admits the following representation (see Barndorff-Nielsen et al. (2013a) and Comte and Renault (1996))

$$Y_t = Y_0 + \int_0^t A_s ds + f(0+) \int_0^t \sigma_s dL_{T_s}, \quad \text{a.s. for } t \geq 0,$$

where

$$A_s := \int_{-\infty}^s f'(s-r) \sigma_r dL_{T_r}, \quad s \in \mathbb{R}.$$

for a suitable function  $f$ . In this case  $Y$  is invertible and  $\sigma$  can be recovered pathwise.

The previous examples showed the importance of the invertibility in the recovery problem for  $\mathcal{LSS}$  processes. This means that in the invertible case, the recovery problem for  $Y$  is analogous to the one of  $X$ . When  $\sigma \equiv 1$ ,  $X$  is just the subordinated Lévy process by  $T$  and in this case the invertibility of  $Y$  holds under very weak conditions.

**Theorem 30** *Suppose that  $Y$  is given as in (27) with  $\sigma \equiv 1$ . Assume in addition that  $f$  is integrable with non-vanishing Fourier transform. Then, for a given Lévy processes  $L$ , the law of  $T$  is completely determined by the law of  $Y$ .*

**Proof.** Let  $L$  a fixed Lévy process with associated law  $\mu \in ID(\mathbb{R})$ . From Theorem 9

$$\mathcal{L}(T_t) = \Lambda_\mu^{-1}[\mathcal{L}(L_{T_t})], \quad t \in \mathbb{R}. \quad (28)$$

Moreover, thanks to Theorem 6 in Sauri (2015),  $Y$  is invertible. Thus, for every  $t \in \mathbb{R}$ , there are two sequences  $(\theta_j^t)_{j=1}^n$  and  $(s_j^t)_{j=1}^n$ , such that

$$\mathcal{L}(L_{T_t}) = \lim_{n \rightarrow \infty} \mathcal{L}\left(\sum_{j=1}^n \theta_j^t Y_{s_j^t}\right),$$

where the limit is taken in the weak sense. Since the law of  $T$  is completely determined by the law of  $T_t$  for some  $t$ , the result follows by inserting the previous equation into (28). ■

**Remark 31** Observe that in general the law of  $Y$  is not uniquely determined by the law of  $T$ . If this was true, it would imply that the mapping  $\mu \mapsto \mathcal{L}\left(\int_{\mathbb{R}} f(s) dL_s^\mu\right)$ , where  $\mu \in ID(\mathbb{R})$  and  $L^\mu$  is the Lévy process associated with  $\mu$ , is one-to-one, which is not true in general.

**Example 32 (Gamma kernel)** Let  $f(x) = e^{-x}x^\alpha$  for  $x \geq 0$  and  $\alpha > -1$ . We have that the corresponding  $\mathcal{LSS}$  process is invertible as Theorem 6 in Sauri (2015) shows. When  $\alpha = 0$ ,  $Y$  is just an Ornstein-Uhlenbeck process and in this case the mapping  $\mu \mapsto \mathcal{L}\left(\int_0^\infty e^{-s} dL_s^\mu\right)$ , where  $\mu \in ID(\mathbb{R})$  and  $L^\mu$  is the Lévy process associated with  $\mu$ , is in a bijection with the subclass of selfdecomposable distributions. Denoting by  $\Psi^0$  such a mapping we see that

$$\mathcal{L}\left(\int_0^\infty e^{-s} dL_{T_s}\right) = \Psi^0 \circ \Lambda(\mu_L, \mu_T),$$

where  $\mu_L \sim L_1$ ,  $\mu_T \sim T_1$ , with  $L$  and  $T$  a Lévy process and  $T$  a subordinator, respectively. In particular, for a fixed Lévy process the mapping  $\Psi^0 \circ \Lambda_{\mu_L}$  is a bijection between the subclass of  $ID(\mathbb{R}^+)$  for which its Lévy measure has log-moments outside of zero, and the class of selfdecomposable distributions. In this case, the law of  $Y$  is uniquely determined by the law of  $T$  and viceversa.

On the other hand, when  $-1 < \alpha < 0$ , the associated  $\mathcal{LSS}$  process is no longer a semimartingale and it is not bounded with positive probability. Moreover, the mapping  $\mu \mapsto \mathcal{L}\left(\int_0^\infty e^{-s} s^\alpha dL_s^\mu\right)$  maps a subclass of  $ID(\mathbb{R})$  into a proper subset of selfdecomposable distributions. As before, let  $\Psi^\alpha$  denote such a transformation, then for a fixed Lévy process  $L$  for which  $\mu_L \in ID(\mathbb{R})$  is its law at time 1, we have that

$$\mathcal{L}\left(\int_0^\infty e^{-s} s^\alpha dL_{T_s}\right) = \Psi^\alpha \circ \Lambda_{\mu_L}(\mu_T).$$

In Pedersen and Sauri (2014), it was shown that under some conditions,  $\Psi^\alpha$  is one-to-one. Therefore, for every  $-1 < \alpha \leq 0$ , the marginal law of the related  $\mathcal{LSS}$  process is uniquely determined by the law of the time change  $T$ .

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